1.2.3 Exercise 8 -

To find an infinite series that is an exception for the rule **(7)**.

If and were finite sequences, rule **(7)** would follow directly from the commutative property of addition, as simply defines the sum of a specific finite set in a specific order of all elements that satisfy both and .

If or defines an infinite set, the meaning of the sum involves then the concept of limit.

The concept of limit comprises the order in which the values of the function are taken into consideration. The order of an infinite sum determines the way the function varies heading to infinity, hence having influence over its limit. Then the order of the sum has a role in determining its value.

Although possible in principle, it seems quite uneasy to find such a sequence. The one Knuth presents is defined by a non-algebraic formula that seems designed from scratch with the solely purpose of solving this problem – what makes me wonder if there’s any algebraic solution for it.

Knuth advances that absolute convergent sums and convergent sums of finite sums satisfy the rule.

His solution is as follows, a non-absolutely convergent series:

, for

The definition of is such that is its inverted sequence. Were the sum over all integers, each internal sum, taking **i** or **j** as leading parameter, its value would be always zero, due to the symmetry of . This asymmetry means that the first internal sum has no pair anymore and will then yield a non-zero value. All other internal sums still have a pair (-1,1) and then yield a zero value.

Uma imagem contendo relógio, pendurado, diferente, quarto

Descrição gerada automaticamente

By taking only non-negative integers (or alternatively only negative), that is a asymmetrical operation, the asymmetry in the relation of **i** and **j** is revealed. As all pairs (1, -1) for become (-1, 1) for , the final values of the sum will be different, and, respectively equal to -1 and 1.

Gráfico, Gráfico de linhas

Descrição gerada automaticamente

Hence,

For what’s worth, I found this solution quite artificial. But it gives a good direction of how to look up or design more sequences that satisfy the rule. It’s about determining a sequence with an asymmetrical relation with respect to **i** and **j,** with being a convergent sequence for both **x = i** and **x = j**. If those conditions are satisfied an asymmetry might arise in the final sum.